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Discriminants and Functional Equations for Polynomials Orthogonal on the Unit Circle

Mourad E. H. Ismail[1]

Department of Mathematics, University of South Florida, Tampa, Florida 33620-5700, U.S.A.

and

Nicholas S. Witte

Department of Mathematics and Statistics and School of Physics, University of Melbourne, Victoria 3010, Australia

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We derive raising and lowering operators for orthogonal polynomials on the unit circle and find second order differential and q-difference equations for these polynomials. A general functional equation is found which allows one to relate the zeros of the orthogonal polynomials to the stationary values of an explicit quasi-energy and implies recurrences on the orthogonal polynomial coefficients. We also evaluate the discriminants and quantized discriminants of polynomials orthogonal on the unit circle. © 2001 Academic Press

Key Words: discriminants; polynomials orthogonal on the unit circle; differential equations; zeros.

1. INTRODUCTION

Let $w(z)$ be a weight function supported on a subset of the unit circle and assume that w is normalized by

$$\int_{|\zeta|=1} w(\zeta)\,\frac{d\zeta}{i\zeta}=1. \tag{1.1}$$

Let $\phi_n(z)$ be the polynomials orthonormal with respect to $w(z)$, that is

$$\int_{|\zeta|=1} \phi_m(\zeta)\,\overline{\phi_n(\zeta)}\,w(\zeta)\,\frac{d\zeta}{i\zeta}=\delta_{m,n}. \tag{1.2}$$

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A general background to orthogonal polynomial systems defined on the unit circle can be found in the monographs [33], [10], and [8], while more recent surveys are to be found in [11], [12] and [24] and from an interesting perspective, in the course notes of [23]. In this work we first derive raising and lowering operators for ϕ_n under certain smoothness conditions on the weight function then use these first order operators to derive a linear second order differential equation satisfied by the orthogonal polynomials. This will be done in Section 2. These results are unit circle analogues of the results of Bauldry [2], Bonan and Clark [4], and Chen and Ismail [5]. The external field [19], [27] is the function v defined by

$$w(z) = \exp(-v(z)). \tag{1.3}$$

We illustrate these general results by three examples—the circular Jacobi polynomials, the Szegő polynomials and the orthogonal polynomial system defined by the modified Bessel function.

Flowing from the results in Section 2 we derive a functional equation and relate this to the zeros of orthogonal polynomials defined on the unit circle in Section 3. This is the analogue of the electrostatic interpretation of the zeros of orthogonal polynomials defined on the real line, but in this case analyticity means that the quasi-energy function derived has stationary points at the zeros which are saddle-points, not minima. This functional equation implies a general relationship on the orthogonal polynomial system, which is usually expressed as a recurrence relation on the polynomial coefficients.

In Section 4 we derive q-analogues of Section 2. The external field is now the function u defined through

$$(D_q w)(z) = -u(qz)\, w(qz), \tag{1.4}$$

where D_q is the q-difference operator

$$(D_q f)(z) := \frac{f(z) - f(qz)}{z - qz}. \tag{1.5}$$

In other words

$$w(z) = w(qz)[1 - (1-q)\, zu(qz)], \qquad |z| = 1. \tag{1.6}$$

Recall that the discriminant $D(f_n)$ of a polynomials f_n is defined by [6]

$$D(f_n) = \gamma^{2n-2} \prod_{1 \leqslant j < k \leqslant n} (z_j - z_k)^2, \qquad \text{if} \quad f_n(z) = \gamma \prod_{j=1}^{n} (z - z_j). \tag{1.7}$$

Stieltjes [30], [31] and Hilbert [15] evaluated the discriminants of the classical orthogonal polynomials of Hermite, Laguerre, and Jacobi. Schur [28] gave an interesting lemma about general orthogonal polynomials which implies the Stieltjes–Hilbert results when applied to the Hermite, Laguerre, and Jacobi polynomials. In Section 4 we prove an analogue of Schur's lemma for polynomials orthogonal on the unit circle and use it to give a general theorem on the evaluation of discriminants of orthogonal polynomials on the unit circle. This is the unit circle analogue of the results in [18]. It was observed in [21] that, in general, the discriminant (1.7) of q-orthogonal polynomials does not have a closed form. The appropriate discriminant for discrete q-orthogonal polynomials is

$$D(f_n, q) = \gamma^{2n-2} q^{n(n-1)/2} \prod_{1 \leqslant j < k \leqslant n} (q^{1/2} z_j - z_k q^{-1/2})(q^{-1/2} z_j - z_k q^{1/2}), \quad (1.8)$$

if f_n is as in (1.7). The above discriminant also has the alternate representation

$$D(f_n, q) = \gamma^{2n-2} q^{n(n-1)/2} \prod_{1 \leqslant j < k \leqslant n} [z_j^2 + z_k^2 - z_j z_k (q + q^{-1})]. \quad (1.9)$$

In particular for a quadratic polynomial $Az^2 + Bz + C$ the q-discriminant is $qB^2 - (1+q)^2 AC$.

In Section 5 we give an expression for the q-discriminant of polynomials orthogonal on the unit circle in terms of the coefficients in the recurrence relations satisfied by the polynomials. As an illustration we evaluate the q-discriminant of the Rogers–Szegő polynomials [32].

2. DIFFERENTIAL EQUATIONS

Recall that if f is a polynomial of degree n then the reciprocal polynomial is

$$f^*(z) := \sum_{k=0}^{n} \overline{a_k} z^{n-k}, \qquad \text{if} \quad f(z) = \sum_{k=0}^{n} a_k z^k, \qquad \text{and} \quad a_n \neq 0, \quad (2.1)$$

[33]. Let ϕ_n satisfy (1.2) and

$$\phi_n(z) = \kappa_n z^n + l_n z^{n-1} + \text{lower order terms}, \qquad \kappa_n > 0 \quad \text{for} \quad n > 0. \quad (2.2)$$

Then the ϕ_n's satisfy the recurrence relations [33, (11.4.6), (11.4.7)]

$$\kappa_n z \phi_n(z) = \kappa_{n+1} \phi_{n+1}(z) - \phi_{n+1}(0)\, \phi_{n+1}^*(z), \quad (2.3)$$

$$\kappa_n \phi_{n+1}(z) = \kappa_{n+1} z \phi_n(z) + \phi_{n+1}(0)\, \phi_n^*(z). \quad (2.4)$$

If we eliminate ϕ_n^* between (2.3) and (2.4) we get the three term recurrence relation (XI.4, p. 91 in [12])

$$\kappa_n \phi_n(0)\, \phi_{n+1}(z) + \kappa_{n-1} \phi_{n+1}(0)\, z\phi_{n-1}(z) = [\kappa_n \phi_{n+1}(0) + \kappa_{n+1} \phi_n(0)\, z]\, \phi_n(z). \tag{2.5}$$

The κ's and $\phi_n(0)$ are related through [33, (11.3.6)]

$$\kappa_n^2 = \sum_{k=0}^{n} |\phi_k(0)|^2. \tag{2.6}$$

Thus κ_n (>0) can be found from the knowledge of $|\phi_k(0)|$. By equating coefficients of z^n in (2.5) and in view of (2.2) we find

$$\kappa_n l_{n+1} \phi_n(0) + \kappa_{n-1}^2 \phi_{n+1}(0) = \kappa_n^2 \phi_{n+1}(0) + \kappa_{n+1} l_n \phi_n(0).$$

Thus

$$\kappa_n l_{n+1} = \kappa_{n+1} l_n + \overline{\phi_n(0)}\, \phi_{n+1}(0). \tag{2.7}$$

From (2.7) it is possible to express l_n in terms of the κ's and $\phi_j(0)$'s

$$l_n = \kappa_n \sum_{j=0}^{n-1} \frac{\overline{\phi_j(0)}\, \phi_{j+1}(0)}{\kappa_j \kappa_{j+1}}. \tag{2.8}$$

The analogue of the Christoffel–Darboux formula is

$$\sum_{k=0}^{n} \overline{\phi_k(a)}\, \phi_k(z) = \frac{\overline{\phi_{n+1}^*(a)}\, \phi_{n+1}^*(z) - \overline{\phi_{n+1}(a)}\, \phi_{n+1}(z)}{1 - \bar{a}z}. \tag{2.9}$$

THEOREM 2.1. *Let $w(z)$ be differentiable in a neighborhood of the unit circle, has moments of all integral orders and assume that the integrals*

$$\int_{|\zeta|=1} \frac{v'(z) - v'(\zeta)}{z - \zeta}\, \zeta^n w(\zeta)\, \frac{d\zeta}{i\zeta}$$

exist for all integers n. Then the corresponding orthonormal polynomials satisfy the differential relation

$$\phi_n'(z) = n \frac{\kappa_{n-1}}{\kappa_n} \phi_{n-1}(z) - i\phi_n^*(z) \int_{|\zeta|=1} \frac{v'(z) - v'(\zeta)}{z - \zeta}\, \phi_n(\zeta)\, \overline{\phi_n^*(\zeta)}\, w(\zeta)\, d\zeta + i\phi_n(z) \int_{|\zeta|=1} \frac{v'(z) - v'(\zeta)}{z - \zeta}\, \phi_n(\zeta)\, \overline{\phi_n(\zeta)}\, w(\zeta)\, d\zeta. \tag{2.10}$$

Proof. Using the orthogonality relation (1.2) we express $\phi_n'(z)$ in terms of the ϕ_k's as

$$\begin{aligned}\phi_n'(z) &= \sum_{k=0}^{n-1} \phi_k(z) \int_{|\zeta|=1} \phi_n'(\zeta)\, \overline{\phi_k(\zeta)}\, w(\zeta)\, \frac{d\zeta}{i\zeta} \\ &= \sum_{k=0}^{n-1} \phi_k(z) \int_{|\zeta|=1} [v'(\zeta)\, \overline{\phi_k(\zeta)} + \overline{\zeta\phi_k(\zeta)} + \overline{\zeta^2\phi_k'(\zeta)}]\, \phi_n(\zeta)\, w(\zeta)\, \frac{d\zeta}{i\zeta},\end{aligned}$$

where we have integrated by parts, then rewritten the derivative of the conjugated polynomial in the following way

$$\frac{d}{d\zeta}\, \overline{\phi_n(\zeta)} = -\bar{\zeta}^2\, \overline{\phi_n'(\zeta)}, \tag{2.11}$$

and used the fact $\bar{\zeta} = 1/\zeta$ for $|\zeta| = 1$. Now the orthogonality relation (1.2) and (2.2) give

$$\begin{aligned}\phi_n'(z) &= \int_{|\zeta|=1} v'(\zeta)\, \phi_n(\zeta) \sum_{k=0}^{n-1} \overline{\phi_k(\zeta)}\, \phi_k(z)\, w(\zeta)\, \frac{d\zeta}{i\zeta} \\ &\quad + \phi_{n-1}(z) \int_{|\zeta|=1} [\overline{\zeta\phi_{n-1}(\zeta)} + \overline{\zeta^2\phi_{n-1}'(\zeta)}]\, \phi_n(\zeta)\, w(\zeta)\, \frac{d\zeta}{i\zeta} \\ &= \int_{|\zeta|=1} \frac{v'(z) - v'(\zeta)}{z-\zeta}\, \phi_n(\zeta) [\overline{\phi_n^*(\zeta)}\, \phi_n^*(z) - \overline{\phi_n(\zeta)}\, \phi_n(z)]\, w(\zeta)\, \frac{d\zeta}{i} \\ &\quad + \phi_{n-1}(z) \left[\frac{\kappa_{n-1}}{\kappa_n} + (n-1)\, \frac{\kappa_{n-1}}{\kappa_n} \right].\end{aligned}$$

This establishes (2.10) and completes the proof.

We next apply (2.3) to eliminate ϕ_n^* from (2.10), assuming $\phi_n(0) \neq 0$. The result is

$$\begin{aligned}\phi_n'(z) &= n\, \frac{\kappa_{n-1}}{\kappa_n}\, \phi_{n-1}(z) + i\, \frac{\kappa_{n-1}}{\phi_n(0)}\, z\phi_{n-1}(z) \\ &\quad \int_{|\zeta|=1} \frac{v'(z) - v'(\zeta)}{z-\zeta}\, \phi_n(\zeta)\, \overline{\phi_n^*(\zeta)}\, w(\zeta)\, d\zeta \\ &\quad + i\phi_n(z) \int_{|\zeta|=1} \frac{v'(z) - v'(\zeta)}{z-\zeta}\, \phi_n(\zeta) \left[\overline{\phi_n(\zeta)} - \frac{\kappa_n}{\phi_n(0)}\, \overline{\phi_n^*(\zeta)} \right] w(\zeta)\, d\zeta.\end{aligned} \tag{2.12}$$

Observe that $\overline{\phi_n(\zeta)} - \frac{\kappa_n}{\phi_n(0)}\, \overline{\phi_n^*(\zeta)}$ is a polynomial of degree $n-1$.

Let

$$A_n(z) = n\frac{\kappa_{n-1}}{\kappa_n} + i\frac{\kappa_{n-1}}{\phi_n(0)} z \int_{|\zeta|=1} \frac{v'(z)-v'(\zeta)}{z-\zeta} \phi_n(\zeta)\, \overline{\phi_n^*(\zeta)}\, w(\zeta)\, d\zeta, \tag{2.13}$$

$$B_n(z) = -i\int_{|\zeta|=1} \frac{v'(z)-v'(\zeta)}{z-\zeta} \phi_n(\zeta) \left[\overline{\phi_n(\zeta)} - \frac{\kappa_n}{\phi_n(0)} \overline{\phi_n^*(\zeta)} \right] w(\zeta)\, d\zeta. \tag{2.14}$$

For future reference we note that $A_0 = B_0 = 0$ and

$$A_1(z) = \kappa_1 - \phi_1(z)\, v'(z) - \frac{\phi_1^2(z)}{\phi_1(0)} M_1(z), \tag{2.15}$$

$$B_1(z) = -v'(z) - \frac{\phi_1(z)}{\phi_1(0)} M_1(z), \tag{2.16}$$

where the first moment M_1 is defined by

$$M_1(z) = \int_{|\zeta|=1} \zeta \frac{v'(z)-v'(\zeta)}{z-\zeta} w(\zeta) \frac{d\zeta}{i\zeta}. \tag{2.17}$$

Now rewrite (2.12) in the form

$$\phi_n'(z) = A_n(z)\, \phi_{n-1}(z) - B_n(z)\, \phi_n(z). \tag{2.18}$$

Define differential operators $L_{n,1}$ and $L_{n,2}$ by

$$L_{n,1} = \frac{d}{dz} + B_n(z), \tag{2.19}$$

$$L_{n,2} = -\frac{d}{dz} - B_{n-1}(z) + \frac{A_{n-1}(z)\,\kappa_{n-1}}{z\kappa_{n-2}} + \frac{A_{n-1}(z)\,\kappa_n \phi_{n-1}(0)}{\kappa_{n-2}\phi_n(0)}. \tag{2.20}$$

After the elimination of ϕ_{n-1} between (2.12) and (2.5) we find that the operators $L_{n,1}$ and $L_{n,2}$ are annihilation and creation operators in the sense that they satisfy

$$L_{n,1}\phi_n(z) = A_n(z)\, \phi_{n-1}(z), \qquad L_{n,2}\phi_{n-1}(z) = \frac{A_{n-1}(z)}{z} \frac{\phi_{n-1}(0)\,\kappa_{n-1}}{\phi_n(0)\,\kappa_{n-2}} \phi_n(z). \tag{2.21}$$

Hence we have established the second order differential equation

$$L_{n,2}\left(\frac{1}{A_n(z)} L_{n,1}\right)\phi_n(z) = \frac{A_{n-1}(z)}{z} \frac{\phi_{n-1}(0)\,\kappa_{n-1}}{\phi_n(0)\,\kappa_{n-2}} \phi_n(z), \tag{2.22}$$

which will also be written in the following way

$$\phi_n'' + P(z)\, \phi_n' + Q(z)\, \phi_n = 0. \tag{2.23}$$

It is worth mentioning that, unlike for polynomials orthogonal on the line, $L^*_{n,1}$ is not related to $L_{n,2}$. In fact if we let

$$(f, g) := \int_{|\zeta|=1} f(\zeta)\,\overline{g(\zeta)}\,w(\zeta)\,\frac{d\zeta}{\zeta}, \tag{2.24}$$

then in the Hilbert space endowed with this inner product, the adjoint of $L_{n,1}$ is

$$(L^*_{n,1} f)(z) = z^2 f'(z) + zf(z) + [\overline{v(z) + B_n(z)}]\, f(z). \tag{2.25}$$

To see this use integration by parts and the fact that for $|\zeta| = 1$, $\overline{g(\zeta)} = \bar{g}(1/\zeta)$.

EXAMPLE 1. The circular Jacobi orthogonal polynomials (CJ) are defined with respect to the weight function

$$w(z) = \frac{\Gamma^2(a+1)}{2\pi\Gamma(2a+1)}\,|1-z|^{2a} \tag{2.26}$$

for real a appropriately restricted. We find these to be classical in the sense of being related to classical orthogonal polynomials defined on the real line and therefore possessing their properties. They arise in a class of random unitary matrix ensembles, the CUE, where the parameter a is related to the charge of an impurity fixed at $z = 1$ in a system of unit charges located on the unit circle at the complex values given by the eigenvalues of a member of this matrix ensemble [35]. The orthonormal polynomials are

$$\phi_n(z) = \frac{(a)_n}{\sqrt{n!\,(2a+1)_n}}\,{}_2F_1(-n, a+1; -n+1-a; z)\,, \tag{2.27}$$

and the coefficients are

$$\kappa_n = \frac{(a+1)_n}{\sqrt{n!\,(2a+1)_n}} \qquad n \geqslant 0, \tag{2.28}$$

$$l_n = \frac{na}{n+a}\,\kappa_n \qquad n \geqslant 1, \tag{2.29}$$

$$\phi_n(0) = \frac{a}{n+a}\,\kappa_n \quad n \geqslant 0. \tag{2.30}$$

The reciprocal polynomials are

$$\phi^*_n(z) = \frac{(a+1)_n}{\sqrt{n!\,(2a+1)_n}}\,{}_2F_1(-n, a; -n-a; z). \tag{2.31}$$

Using the differentiation formula and some contiguous relations for the hypergeometric functions, combined in the form

$$(1-z)\frac{d}{dz}\,{}_2F_1(-n, a+1; 1-n-a; z)$$
$$=\frac{n(n+2a)}{n-1+a}\,{}_2F_1(1-n, a+1; 2-n-a; z)$$
$$-n\,{}_2F_1(-n, a+1; 1-n-a; z), \tag{2.32}$$

one finds the differential-recurrence relation

$$(1-z)\,\phi_n' = -n\phi_n + [n(n+2a)]^{1/2}\,\phi_{n-1}, \tag{2.33}$$

and the coefficient functions

$$A_n(z) = \frac{\sqrt{n(n+2a)}}{1-z}, \tag{2.34}$$

$$B_n(z) = \frac{n}{1-z}. \tag{2.35}$$

The second order differential equation becomes

$$\phi_n'' + \phi_n' \left\{\frac{1-n-a}{z} - \frac{2a+1}{1-z}\right\} + \phi_n \frac{n(a+1)}{z(1-z)} = 0. \tag{2.36}$$

EXAMPLE 2. We consider a generalization of the previous example, to the situation where

$$w(z) = 2^{-1-2a-2b}\frac{\Gamma(a+b+1)}{\Gamma(a+1/2)\,\Gamma(b+1/2)}\,|1-z|^{2a}\,|1+z|^{2b}, \tag{2.37}$$

with $x = \cos\theta$, and the associated orthogonal polynomials are known as Szegő polynomials [33]. They are related to the Jacobi polynomials via the projective mapping of the unit circle onto the interval $[-1, 1]$, $z \mapsto \frac{1}{2}[z+z^{-1}] = x = \cos\theta$,

$$z^{-n}\phi_{2n}(z) = \mathbb{A}P_n^{(a-1/2,\,b-1/2)}(\tfrac{1}{2}[z+z^{-1}])$$
$$+\tfrac{1}{2}\mathbb{B}[z-z^{-1}]\,P_{n-1}^{(a+1/2,\,b+1/2)}(\tfrac{1}{2}[z+z^{-1}]), \tag{2.38}$$
$$z^{1-n}\phi_{2n-1}(z) = \mathbb{C}P_n^{(a-1/2,\,b-1/2)}(\tfrac{1}{2}[z+z^{-1}])$$
$$+\tfrac{1}{2}\mathbb{D}[z-z^{-1}]\,P_{n-1}^{(a+1/2,\,b+1/2)}(\tfrac{1}{2}[z+z^{-1}]). \tag{2.39}$$

In their study of the equilibrium positions of charges confined to the unit circle subject to logarithmic repulsion Forrester and Rogers considered orthogonal polynomials defined on x which are just the first term of 2.38.

Using the normalization amongst the even and odd sequences of polynomials, orthogonality between these two sequences and the requirement that the coefficient of z^{-n} on the right-hand side of (2.39) must vanish, one finds explicitly that the coefficients are

$$\mathbb{A} = \left\{ \frac{n!\,(a+b+1)_n}{(a+1/2)_n\,(b+1/2)_n} \right\}^{1/2}, \tag{2.40}$$

$$\mathbb{B} = \tfrac{1}{2}\mathbb{A}, \tag{2.41}$$

$$\mathbb{C} = n \left\{ \frac{(n-1)!\,(a+b+1)_{n-1}}{(a+1/2)_n\,(b+1/2)_n} \right\}^{1/2}, \tag{2.42}$$

$$\mathbb{D} = \frac{n+a+b}{2n}\,\mathbb{C}. \tag{2.43}$$

Furthermore the following coefficients of the polynomials are found to be

$$\kappa_{2n} = 2^{-2n} \frac{(a+b+1)_{2n}}{\sqrt{n!\,(a+b+1)_n\,(a+1/2)_n\,(b+1/2)_n}}, \tag{2.44}$$

$$\kappa_{2n-1} = 2^{1-2n} \frac{(a+b+1)_{2n-1}}{\sqrt{(n-1)!\,(a+b+1)_{n-1}\,(a+1/2)_n\,(b+1/2)_n}}, \tag{2.45}$$

$$l_{2n} = 2n \frac{a-b}{2n+a+b}\,\kappa_{2n}, \tag{2.46}$$

$$l_{2n-1} = (2n-1) \frac{a-b}{2n+a+b-1}\,\kappa_{2n-1}, \tag{2.47}$$

$$\phi_{2n}(0) = \frac{a+b}{2n+a+b}\,\kappa_{2n}, \tag{2.48}$$

$$\phi_{2n-1}(0) = \frac{a-b}{2n+a+b-1}\,\kappa_{2n-1}. \tag{2.49}$$

The three term recurrences are then

$$\begin{aligned} &2(a-b)\sqrt{n(n+a+b)}\,\phi_{2n}(z) \\ &\quad + 2(a+b)\sqrt{(n+a-1/2)(n+b-1/2)}\,z\phi_{2n-2}(z) \\ &\quad = [(a+b)(2n+a+b-1) + (a-b)(2n+a+b)\,z]\,\phi_{2n-1}(z), \end{aligned} \tag{2.50}$$

and

$$\begin{aligned} &2(a+b)\sqrt{(n+a-1/2)(n+b-1/2)}\,\phi_{2n-1}(z) \\ &\quad + 2(a-b)\sqrt{(n-1)(n+a+b-1)}\,z\phi_{2n-3}(z) \\ &\quad = [(a-b)(2n+a+b-2) + (a+b)(2n+a+b-1)\,z]\,\phi_{2n-2}(z), \end{aligned} \tag{2.51}$$

when $a \neq b$ and both these degenerate to $\phi_{2n-1}(z) = z\phi_{2n-2}(z)$ when $a = b$. For reference the reciprocal polynomials are

$$z^{-n}\phi^*_{2n}(z) = \mathbb{A}P_n^{(a-1/2,\,b-1/2)}(\tfrac{1}{2}[z+z^{-1}]) - \tfrac{1}{2}\,\mathbb{B}[z-z^{-1}]\,P_{n-1}^{(a+1/2,\,b+1/2)}(\tfrac{1}{2}[z+z^{-1}]), \quad (2.52)$$

$$z^{1-n}\phi^*_{2n-1}(z) = \mathbb{C}zP_n^{(a-1/2,\,b-1/2)}(\tfrac{1}{2}[z+z^{-1}]) - \tfrac{1}{2}\,\mathbb{D}[z-z^{-1}]\,zP_{n-1}^{(a+1/2,\,b+1/2)}(\tfrac{1}{2}[z+z^{-1}]). \quad (2.53)$$

Using the differential and recurrence relations for the Jacobi polynomials directly one can find the appropriate coefficient functions for the Szegő polynomials to be

$$A_{2n-1}(z) = 2\sqrt{(n+a-1/2)(n+b-1/2)}\;\frac{a-b+(a+b)\,z}{(a-b)(1-z^2)}, \quad (2.54)$$

$$B_{2n-1}(z) = \frac{4ab+(2n-1)[a+b+(a-b)\,z]}{(a-b)(1-z^2)}, \quad (2.55)$$

$$A_{2n}(z) = 2\sqrt{n(n+a+b)}\;\frac{a+b+(a-b)\,z}{(a+b)(1-z^2)}, \quad (2.56)$$

$$B_{2n}(z) = 2n\,\frac{a-b+(a+b)\,z}{(a+b)(1-z^2)}, \quad (2.57)$$

again when $a \neq b$.

EXAMPLE 3. Consider the weight function

$$w(z) = \frac{1}{2\pi I_0(t)}\exp(\tfrac{1}{2}t[z+z^{-1}]), \quad (2.58)$$

where I_ν is a modified Bessel function. This system of orthogonal polynomials has arisen from studies of the length of longest increasing subsequences of random words [3] and matrix models [26], [16]. In terms of the leading coefficient one has the Toeplitz determinant form

$$\kappa_n^2(t) = I_0(t)\,\frac{\det(I_{j-k}(t))_{0\leqslant j,\,k\leqslant n-1}}{\det(I_{j-k}(t))_{0\leqslant j,\,k\leqslant n}}. \quad (2.59)$$

The first few members of this sequence are

$$\kappa_1^2 = \frac{I_0^2(t)}{I_0^2(t)-I_1^2(t)}, \quad (2.60)$$

$$\frac{\phi_1(0)}{\kappa_1} = -\frac{I_1(t)}{I_0(t)}, \quad (2.61)$$

$$\kappa_2^2 = \frac{I_0(t)(I_0^2(t)-I_1^2(t))}{(I_0(t)-I_2(t))[I_0(t)(I_0(t)+I_2(t))-2I_1^2(t)]}, \quad (2.62)$$

$$\frac{\phi_2(0)}{\kappa_2} = \frac{I_0(t)\,I_2(t)-I_1^2(t)}{I_1^2(t)-I_0^2(t)}. \quad (2.63)$$

Gessel [13] has found the exact power series expansions in t for the first three determinants which appear in the above coefficients. Some recurrence relations for the corresponding coefficients of the monic version of these orthogonal polynomials have been known [26], [16], [34] and we derive the equivalent results for κ_n, etc.

LEMMA 2.2 [26]. *The reflection coefficient $r_n(t) \equiv \phi_n(0)/\kappa_n$ for the modified Bessel orthogonal polynomial system satisfies a form of the discrete Painlève II equation, namely the recurrence relation*

$$-2\frac{n}{t}\frac{r_n}{1-r_n^2} = r_{n+1} + r_{n-1}, \tag{2.64}$$

for $n \geqslant 1$ and $r_0(t) = 1$, $r_1(t) = -I_1(t)/I_0(t)$.

Proof. First we make a slight redefinition of the external field $w(z) = \exp(-v(z+1/z))$ for convenience. Employing integration by parts we evaluate

$$\begin{aligned} &-\int v'(\zeta+1/\zeta)(1-1/\zeta^2)\,\phi_{n+1}(\zeta)\,\overline{\phi_n(\zeta)}\,w(\zeta)\,\frac{d\zeta}{i\zeta} \\ &\quad = \int [\phi_{n+1}(\zeta)\bar{\zeta}^2\,\overline{\phi_n'(\zeta)} + \phi_{n+1}(\zeta)\,\bar{\zeta}\,\overline{\phi_n(\zeta)} - \phi_{n+1}'(\zeta)\,\overline{\phi_n(\zeta)}]\,w(\zeta)\,\frac{d\zeta}{i\zeta} \\ &\quad = (n+1)\left[\frac{\kappa_n}{\kappa_{n+1}} - \frac{\kappa_{n+1}}{\kappa_n}\right], \end{aligned} \tag{2.65}$$

for general external fields $v(z)$ using (1.2) and (2.2) in a similar way to the proof of Theorem 2.1. However in this case $v'(\zeta+1/\zeta) = -t/2$, a direct evaluation of the left-hand side yields

$$-\frac{1}{2}t\left(\frac{l_n}{\kappa_{n+1}} - \frac{\kappa_n l_{n+2}}{\kappa_{n+1}\kappa_{n+2}}\right), \tag{2.66}$$

and simplification of this equality in terms of the defined ratio and use of (2.8) gives the above result.

There is also a differential relation satisfied by these coefficient functions or equivalently a differential relation in t for the orthogonal polynomials themselves [16], [34].

LEMMA 2.3. *The modified Bessel orthogonal polynomials satisfy the differential relation*

$$\begin{aligned} 2\frac{d}{dt}\phi_n(z) &= \left[\frac{I_1(t)}{I_0(t)} + \frac{\phi_{n+1}(0)}{\kappa_{n+1}}\frac{\kappa_n}{\phi_n(0)}\right]\phi_n(z) \\ &\quad -\frac{\kappa_{n-1}}{\kappa_n}\left[1 + \frac{\phi_{n+1}(0)}{\kappa_{n+1}}\frac{\kappa_n}{\phi_n(0)}z\right]\phi_{n-1}(z), \end{aligned} \tag{2.67}$$

for $n \geqslant 1$ *and* $\frac{d}{dt}\phi_0(z) = 0$. *The differential equations for the coefficients are*

$$\frac{2}{\kappa_n}\frac{d}{dt}\kappa_n = \frac{I_1(t)}{I_0(t)} + \frac{\phi_{n+1}(0)}{\kappa_{n+1}}\frac{\phi_n(0)}{\kappa_n}, \tag{2.68}$$

$$\frac{2}{\phi_n(0)}\frac{d}{dt}\phi_n(0) = \frac{I_1(t)}{I_0(t)} + \frac{\phi_{n+1}(0)}{\kappa_{n+1}}\frac{\kappa_n}{\phi_n(0)} - \frac{\phi_{n-1}(0)}{\phi_n(0)}\frac{\kappa_{n-1}}{\kappa_n}, \tag{2.69}$$

for $n \geqslant 1$.

Proof. Differentiating the orthonormality relation (1.2) with respect to t one finds from the orthogonality principle for $m \leqslant n-2$ that

$$\frac{d}{dt}\phi_n(z) + \tfrac{1}{2}z\phi_n(z) = a_n\phi_{n+1}(z) + b_n\phi_n(z) + c_n\phi_{n-1}(z) \tag{2.70}$$

for some coefficients a_n, b_n, c_n. The first coefficient is immediately found to be $a_n = \frac{1}{2}\kappa_n/\kappa_{n+1}$. Consideration of the differentiated orthonormality relation for $m = n-1$ sets another coefficient, $c_n = -\frac{1}{2}\kappa_{n-1}/\kappa_n$, while the case of $m = n$ leads to $b_n = \frac{1}{2}I_1(t)/I_0(t)$. Finally use of the three-term recurrence (2.5) allows one to eliminate $\phi_{n+1}(z)$ in favor of $\phi_n(z), \phi_{n-1}(z)$ and one arrives at (2.67). The differential equations for the coefficients $\kappa_n, \phi_n(0)$ in (2.68, 2.69) follow from reading off the appropriate terms of (2.67).

Use of the recurrence relation and the differential relations will allow us to find a differential equation for the coefficients, and thus another characterization of the coefficients.

LEMMA 2.4. *The reflection coefficient* $r_n(t)$ *satisfies the following second order differential equation*

$$\frac{d^2}{dt^2}r_n = \frac{1}{2}\left(\frac{1}{r_n+1} + \frac{1}{r_n-1}\right)\left(\frac{d}{dt}r_n\right)^2 - \frac{1}{t}\frac{d}{dt}r_n - r_n(1-r_n^2) + \frac{n^2}{t^2}\frac{r_n}{1-r_n^2}, \tag{2.71}$$

with the boundary conditions determined by the expansion

$$r_n(t) \underset{t\to 0}{\sim} \frac{(-\frac{1}{2}t)^n}{n!}\left\{1 + \left(\frac{n}{n+1} - \delta_{n1}\right)\frac{1}{4}t^2 + \mathrm{O}(t^4)\right\}, \tag{2.72}$$

for $n \geqslant 1$. *The coefficient* r_n *is related by*

$$r_n(t) = \frac{z_n(t)+1}{z_n(t)-1}, \tag{2.73}$$

to $z_n(t)$ *which satisfies the Painléve transcendent P-V equation with the parameters*

$$\alpha = \beta = \frac{n^2}{8}, \qquad \gamma = 0, \qquad \delta = -2. \tag{2.74}$$

Proof. Subtracting the relations (2.68, 2.69) leads to the simplified expression

$$r_{n+1}-r_{n-1}=\frac{2}{1-r_n^2}\frac{d}{dt}r_n, \tag{2.75}$$

which should be compared to the recurrence relation, in a similar form

$$r_{n+1}+r_{n-1}=-2\frac{n}{t}\frac{r_n}{1-r_n^2}. \tag{2.76}$$

The differential equation (2.71) is found by combining these latter two equations and the identification with the P-V can be easily verified.

As a consequence of the above we find that the coefficients for the modified Bessel orthogonal polynomials can be determined by the Toeplitz determinant (2.59), by the recurrence relations (2.76) or by the differential equation (2.71). An example of the use of this last method we note

$$\kappa_n^2(t)=I_0(t)\,[1-r_n^2(t)]^{-1/2}\exp\left(-n\int_0^t\frac{r_n^2(s)}{1-r_n^2(s)}\frac{ds}{s}\right). \tag{2.77}$$

We now indicate how to find the coefficients of the differential relations, $A_n(z)$, $B_n(z)$ and observe that

$$\frac{v'(z)-v'(\zeta)}{z-\zeta}=-\frac{t}{2}\left[\frac{1}{z\zeta^2}+\frac{1}{z^2\zeta}\right].$$

The above relationship and (2.12) yield

$$\begin{aligned}\phi_n'(z)=&\frac{\kappa_{n-1}}{\kappa_n}\phi_{n-1}(z)\left[n+\frac{t}{2z}\right]\\&+\frac{t}{2}\frac{\kappa_{n-1}}{\phi_n(0)}\phi_{n-1}(z)\int_{|\zeta|=1}\phi_n(\zeta)\,\overline{\zeta\phi_n^*(\zeta)}\,w(\zeta)\frac{d\zeta}{i\zeta}\\&+\frac{t}{2z}\phi_n(z)\int_{|\zeta|=1}\phi_n(\zeta)\,\overline{\zeta}\left[\overline{\phi_n(\zeta)}-\frac{\kappa_n}{\phi_n(0)}\overline{\phi_n^*(\zeta)}\right]w(\zeta)\frac{d\zeta}{i\zeta}.\end{aligned} \tag{2.78}$$

Easy calculations using (2.2) give

$$\begin{aligned}&\zeta\left[\phi_n(\zeta)-\frac{\kappa_n}{\overline{\phi_n(0)}}\phi_n^*(\zeta)\right]\\&\quad=-\frac{\kappa_{n-1}}{\kappa_n}\frac{\overline{\phi_{n-1}(0)}}{\overline{\phi_n(0)}}\phi_n(\zeta)+\text{lower order terms}\end{aligned}$$

and

$$\zeta \frac{\phi_n^*(\zeta)}{\phi_n(0)} = \frac{\phi_{n+1}(\zeta)}{\kappa_{n+1}} + \left[\frac{\kappa_n l_n - \kappa_{n-1} l_{n-1}}{\kappa_n |\phi_n(0)|^2} - \frac{l_{n+1}}{k_{n+1}\kappa_n} \right] \phi_n(\zeta) + \text{lower order terms.}$$

These identities together with (2.79) establish the differential-difference relation

$$\phi_n'(z) = \frac{\kappa_{n-1}}{\kappa_n} \left[n + \frac{t}{2z} + \frac{t}{2} \frac{\kappa_{n-1}}{\kappa_n} \frac{\phi_{n-1}(0)}{\phi_n(0)} - \frac{t}{2} \frac{\overline{\phi_{n+1}(0)}\, \phi_n(0)}{\kappa_{n+1}\kappa_n} \right] \phi_{n-1}(z) - \frac{t}{2z} \frac{\kappa_{n-1}}{\kappa_n} \frac{\phi_{n-1}(0)}{\phi_n(0)} \phi_n(z). \tag{2.79}$$

3. FUNCTIONAL EQUATION AND ZEROS

In this section we continue the development of the previous discussion of the differential relations satisfied by orthogonal polynomials to find a functional equation and its relationship to the zeros of the polynomials. Expressing the second order differential equation (2.22) in terms of the coefficient functions $A_n(z)$ and $B_n(z)$ we have

$$\phi_n'' + \left\{ B_n + B_{n-1} - A_n'/A_n - \frac{\kappa_{n-1}}{\kappa_{n-2}} \frac{A_{n-1}}{z} - \frac{\kappa_n}{\kappa_{n-2}} \frac{\phi_{n-1}(0)}{\phi_n(0)} A_{n-1} \right\} \phi_n' + \left\{ B_n' - B_n A_n'/A_n + B_n B_{n-1} - \frac{\kappa_{n-1}}{\kappa_{n-2}} \frac{A_{n-1} B_n}{z} - \frac{\kappa_n}{\kappa_{n-2}} \frac{\phi_{n-1}(0)}{\phi_n(0)} A_{n-1} B_n + \frac{\kappa_{n-1}}{\kappa_{n-2}} \frac{\phi_{n-1}(0)}{\phi_n(0)} \frac{A_{n-1} A_n}{z} \right\} \phi_n = 0. \tag{3.1}$$

Now by analogy with the orthogonal polynomials defined on the real line the coefficient of the ϕ_n' term above can be simplified.

THEOREM 3.1. *Given that $v(z)$ is an meromorphic function in the unit disk then the following functional equation holds*

$$B_n + B_{n-1} - \frac{\kappa_{n-1}}{\kappa_{n-2}} \frac{A_{n-1}}{z} - \frac{\kappa_n}{\kappa_{n-2}} \frac{\phi_{n-1}(0)}{\phi_n(0)} A_{n-1} = -(n-1)\, z^{-1} - v'(z). \tag{3.2}$$

Proof. From the definitions (2.13, 2.14) we start with the following expression

$$
\begin{aligned}
B_n + B_{n-1} - \frac{\kappa_{n-1}}{\kappa_{n-2}} \frac{A_{n-1}}{z} - \frac{\kappa_n}{\kappa_{n-2}} \frac{\phi_{n-1}(0)}{\phi_n(0)} A_{n-1} \\
= -(n-1) \left[\frac{1}{z} + \frac{\kappa_n}{\kappa_{n-1}} \frac{\phi_{n-1}(0)}{\phi_n(0)} \right] \\
+ i \int \frac{v'(z) - v'(\zeta)}{z - \zeta} \left\{ -\phi_n \overline{\phi_n} + \frac{\kappa_n}{\phi_n(0)} \phi_n \overline{\phi_n^*} - \phi_{n-1} \overline{\phi_{n-1}} \right. \\
\left. - \frac{\kappa_n}{\phi_n(0)} \zeta \phi_{n-1} \overline{\phi_{n-1}^*} \right\} w(\zeta)\, d\zeta \\
- i \frac{\kappa_n}{\phi_n(0)} \int [v'(z) - v'(\zeta)]\, \phi_{n-1} \overline{\phi_{n-1}^*} w(\zeta)\, d\zeta.
\end{aligned}
$$

Employing the recurrences (2.4, 2.3), and the relation amongst coefficients (2.6) one can show that the factor in the first integral on the right-hand side above is

$$
-\phi_n \overline{\phi_n} + \frac{\kappa_n}{\phi_n(0)} \phi_n \overline{\phi_n^*} - \phi_{n-1} \overline{\phi_{n-1}} - \frac{\kappa_n}{\phi_n(0)} \zeta \phi_{n-1} \overline{\phi_{n-1}^*} = -\phi_n \overline{\phi_n} + \phi_n^* \overline{\phi_n^*}.
$$

Now since $|\zeta|^2 = 1$, one can show that the right-hand side of the above is zero from the Christoffel–Darboux sum (2.9). Consequently our right-hand side is now

$$
\begin{aligned}
-(n-1) \left[\frac{1}{z} + \frac{\kappa_n}{\kappa_{n-1}} \frac{\phi_{n-1}(0)}{\phi_n(0)} \right] \\
- i \frac{\kappa_n}{\phi_n(0)} \left[v'(z) \int \phi_{n-1} \overline{\phi_{n-1}^*} w(\zeta)\, d\zeta - \int v'(\zeta)\, \phi_{n-1} \overline{\phi_{n-1}^*} w(\zeta)\, d\zeta \right].
\end{aligned}
$$

Taking the first integral in this expression and using the recurrence (2.4) and the decomposition $\zeta \phi_{n-1} = \kappa_{n-1}/\kappa_n \phi_n + \pi_{n-1}$ where $\pi_n \in \Pi_n$, Π_n being the space of polynomials of degree at most n, we find it reduces to $-i\phi_n(0)/\kappa_n$ from the normality of the orthogonal polynomials. Considering now the second integral above we integrate by parts and are left with

$$
\int \phi_{n-1}' \overline{\phi_{n-1}^*} w(\zeta)\, d\zeta + \int \phi_{n-1} \overline{\phi_{n-1}^*}' w(\zeta)\, d\zeta,
$$

and the first term here must vanish as ϕ^*_{n-1} can be expressed in terms of ϕ_{n-1}, ϕ_n from (2.4) but $\phi'_{n-1} \in \Pi_{n-2}$. The remaining integral, the second one above, can be treated in the following way. Firstly express the conjugate polynomial in terms of the polynomial itself via (2.3) and employ the relation for its derivative (2.11). Further noting that $\zeta\phi'_{n-1} = (n-1)\,\phi_{n-1} + \pi_{n-2}$, $\zeta\phi_{n-2} = \kappa_{n-2}/\kappa_{n-1}\phi_{n-1} + \pi_{n-2}$, and $\zeta^2\phi'_{n-2} = (n-2)\,\kappa_{n-2}/\kappa_{n-1}\phi_{n-1} + \pi_{n-2}$ along with the orthonormality relation, the final integral is nothing but $-i(n-1)\,\phi_{n-1}(0)/\kappa_{n-1}$. Combining all this the final result is (3.2).

Remark 1. The zeros of the orthogonal polynomial $\phi_n(z)$ are denoted by $\{z_j\}_{1 \leqslant j \leqslant n}$, and are confined to the convex Hull of the support of the measure, namely to be strictly confined within the unit circle $|z| < 1$. One can construct a real function $|T(z_1, ..., z_n)|$ from

$$T(z_1, ..., z_n) = \prod_{j=1}^{n} z_j^{-n+1}\, \frac{e^{-v(z_j)}}{A_n(z_j)} \prod_{1 \leqslant j < k \leqslant n} (z_j - z_k)^2, \tag{3.3}$$

such that the zeros are given by the stationary points of this function. One might also interpret this function as a total energy function for n mobile unit charges in the unit disk interacting with a one-body confining potential, $v(z) + \ln A_n(z)$, an attractive logarithmic potential with a charge $n-1$ at the origin, $(n-1)\ln z$, and repulsive logarithmic two-body potentials, $-\ln(z_i - z_j)$, between pairs of charges. However all the stationary points are saddle-points, a natural consequence of analyticity in the unit disk. That such this function exhibits stationary properties at the zeros can be seen by considering the second order differential equation which in view of the above theorem has the coefficient $P(z)$, namely

$$P(z) = -(n-1)\, z^{-1} - v'(z) - A'_n/A_n. \tag{3.4}$$

This function is a perfect differential and consequently the one-body potential can be constructed from its integral, via the Stieltjes argument. Or alternatively one can show that the conditions for the stationary points of function $T(z_1, ..., z_n)$ above lead to a system of equations

$$-v'(z_j) - \frac{A'_n(z_j)}{A_n(z_j)} - \frac{n-1}{z_j} + 2 \sum_{1 \leqslant k \leqslant n,\, k \neq j} \frac{1}{z_j - z_k} = 0 \qquad j = 1, ..., n. \tag{3.5}$$

Then the pairwise sum can be represented in terms of the polynomial $f(z) = \prod_{j=1}^{n}(z - z_j)$ thus

$$2 \sum_{1 \leqslant k \leqslant n,\, k \neq j} \frac{1}{z_j - z_k} = \frac{f''(z_j)}{f'(z_j)}, \tag{3.6}$$

and we have the n conditions expressed as

$$f''(z_j) + \left\{ -\frac{n-1}{z_j} - v'(z_j) - \frac{A_n'(z_j)}{A_n(z_j)} \right\} f'(z_j) = 0, \qquad \forall j = 1, ..., n. \quad (3.7)$$

The result then follows.

Remark 2. The functional equation (3.2) actually implies a very general recurrence relation on the orthogonal system coefficients κ_n, $\phi_n(0)$. In general if it is possible to relate the differential recurrence coefficients A_n, B_n to these polynomial coefficients, then the functional equation dictates that equality holds for all z, and thus for independent terms in z. For rational functions this can be applied to the coefficients of monomials in z.

Remark 3. There is another way of deriving the functional equation (3.2) which we now describe. Equation (2.22) is one way of expressing the second order differential equation for the orthogonal polynomials, however one can perform the elimination in the opposite order and find

$$L_{n+1,1}\left(\frac{z}{A_n(z)} L_{n+1,2}\right) \phi_n(z) = \frac{\kappa_n \phi_n(0)}{\kappa_{n-1}\phi_{n+1}(0)} A_{n+1}(z)\, \phi_n(z), \quad (3.8)$$

which written out in full is

$$\begin{aligned} \phi_n'' + \left\{ B_{n+1} + B_n - A_n'/A_n - \frac{\kappa_n}{\kappa_{n-1}} \frac{A_n}{z} - \frac{\kappa_{n+1}}{\kappa_{n-1}} \frac{\phi_n(0)}{\phi_{n+1}(0)} A_n + \frac{1}{z} \right\} \phi_n' \\ + \left\{ B_n' - B_n A_n'/A_n + B_{n+1} B_n - \frac{\kappa_n}{\kappa_{n-1}} \frac{A_n B_{n+1}}{z} \right. \\ - \frac{\kappa_{n+1}}{\kappa_{n-1}} \frac{\phi_n(0)}{\phi_{n+1}(0)} A_n B_{n+1} + \frac{\kappa_n}{\kappa_{n-1}} \frac{\phi_n(0)}{\phi_{n+1}(0)} \frac{A_n A_{n+1}}{z} \\ \left. + \frac{B_n}{z} - \frac{\kappa_{n+1}}{\kappa_{n-1}} \frac{\phi_n(0)}{\phi_{n+1}(0)} \frac{A_n}{z} \right\} \phi_n = 0. \end{aligned} \quad (3.9)$$

Given that the coefficient $P(z)$ for these two forms (3.1, 3.9) must be identical we have an inhomogeneous first order difference equation, whose solution is

$$\begin{aligned} B_n + B_{n-1} - \frac{\kappa_{n-1}}{\kappa_{n-2}} \frac{A_{n-1}}{z} - \frac{\kappa_n}{\kappa_{n-2}} \frac{\phi_{n-1}(0)}{\phi_n(0)} A_{n-1} \\ = -(n-1)\, z^{-1} + \text{function of } z \text{ only}. \end{aligned} \quad (3.10)$$

This function can be simply evaluated by setting $n = 1$, evaluating the integrals after noting $B_0 = 0$ and the cancellations, and yields the result $-v'(z)$.

EXAMPLE 1. We can verify that the general form for the T-function is correct in the case of the circular Jacobi polynomials by a direct evaluation

$$T(z_1, ..., z_n) = \prod_{j=1}^{n} z_j^{1-n-a}(1-z_j)^{a+1}(z_j-1)^a \prod_{1 \leqslant j<k \leqslant n} (z_j - z_k)^2, \tag{3.11}$$

where we have used the identity

$$|1-z|^{2a} = (1-z)^a (1-1/z)^a = z^{-a}(1-z)^a (z-1)^a, \tag{3.12}$$

on $|z| = 1$ to suitably construct a locally analytic weight function. One can show that the stationary points for this problem are the solution to the set of equations

$$\frac{1-n-a}{z_j} - \frac{2a+1}{1-z_j} + 2 \sum_{j \neq k} \frac{1}{z_j - z_k} = 0, \qquad 1 \leqslant j \leqslant n, \tag{3.13}$$

so that the polynomial $f(z) = \prod_{j=1}^{n}(z - z_j)$ satisfies the relations

$$f''(z_j) + f'(z_j) \left\{ \frac{1-n-a}{z_j} - \frac{2a+1}{1-z_j} \right\} = 0. \tag{3.14}$$

Consequently we find that

$$z(1-z) f''(z) + f'(z)\{(1-n-a)(1-z) - (2a+1) z\} + Q f(z) = 0, \tag{3.15}$$

for some constant Q independent of z, but possibly dependent on n and a and is identical to the second order ODE (2.36).

EXAMPLE 2. Using the expressions (2.54–2.57) one can verify that the identity (3.2) holds and in particular becomes

$$\begin{aligned} B_n + B_{n-1} - \frac{\kappa_{n-1}}{\kappa_{n-2}} \frac{A_{n-1}}{z} - \frac{\kappa_n}{\kappa_{n-2}} \frac{\phi_{n-1}(0)}{\phi_n(0)} A_{n-1} \\ = -\frac{n-1}{z} - \frac{a+b}{z} - \frac{2a}{1-z} + \frac{2b}{1+z}, \end{aligned} \tag{3.16}$$

for both the odd and even sequences. Consequently the coefficients in the second order differential equation are

$$\begin{aligned} P_n(z) &= -\frac{n+a+b-1}{z} - \frac{2a+1}{1-z} + \frac{2b+1}{1+z} - \frac{a \pm b}{a \mp b + (a \pm b) z}, \\ Q_{2n}(z) &= 2n \frac{a(a+1)(1+z)^2 - b(b+1)(1-z)^2}{z(1-z^2)[a+b+(a-b) z]}, \end{aligned} \tag{3.17}$$

$$Q_{2n-1}(z) = \frac{(2n-1)[a(a+1)(1+z)^2 + b(b+1)(1-z)^2 - 2ab(1-z^2)] + 4ab}{z(1-z^2)[a-b+(a+b) z]}. \tag{3.18}$$

Similarly we can verify that the general form for the T-function is correct in the case of the Szegö polynomials by using the identity

$$|1-z|^{2a}\,|1+z|^{2b} = z^{-a-b}(1-z)^a\,(z-1)^a\,(1+z)^{2b}, \tag{3.20}$$

to suitably analytically continue the weight function. The stationary points for this problem are the solution to the following set of equations

$$\frac{1-n-a-b}{z_j} - \frac{2a+1}{1-z_j} + \frac{2b+1}{1+z_j} - \frac{a \mp b}{a \pm b + (a \mp b)\, z_j} + 2 \sum_{j \neq k} \frac{1}{z_j - z_k} = 0, \qquad 1 \leqslant j \leqslant n, \tag{3.21}$$

such that the polynomial $f(z) = \prod_{j=1}^{n}(z - z_j)$ satisfies the relations

$$f''(z_j) + f'(z_j) \left\{ \frac{1-n-a-b}{z_j} - \frac{2a+1}{1-z_j} + \frac{2b+1}{1+z_j} - \frac{a \mp b}{a \pm b + (a \mp b)\, z_j} \right\} = 0. \tag{3.22}$$

Finally we find that

$$f''(z) + f'(z) \left\{ \frac{1-n-a-b}{z} - \frac{2a+1}{1-z} + \frac{2b+1}{1+z} - \frac{a \mp b}{a \pm b + (a \mp b)\, z} \right\} + Q(z)\, f(z) = 0, \tag{3.23}$$

for some constant Q independent of z, but possibly dependent on n and a. The coefficient of the first derivative term is identical to the expression for $P(z)$ in (3.17).

EXAMPLE 3. One can also verify the functional relation (3.2) for the modified Bessel orthogonal polynomials. Forming the left-hand side of this identity we find this reduces to

$$\begin{aligned} B_n + B_{n-1} - \frac{\kappa_{n-1}}{\kappa_{n-2}} \frac{A_{n-1}}{z} - \frac{\kappa_n}{\kappa_{n-2}} \frac{\phi_{n-1}(0)}{\phi_n(0)} A_{n-1} &= -\frac{n-1}{z} - \frac{t}{2z^2} - (n-1) \frac{\kappa_n}{\kappa_{n-1}} \frac{\phi_{n-1}(0)}{\phi_n(0)} \\ &\quad - \frac{t}{2} \frac{\kappa_n \kappa_{n-2}}{\kappa_{n-1}^2} \frac{\phi_{n-2}(0)}{\phi_n(0)} + \frac{t}{2} \frac{\phi_{n-1}^2(0)}{\kappa_{n-1}^2}. \end{aligned} \tag{3.24}$$

Now the last three terms on the right-hand side of the above equation simplify to $t/2$ using the recurrence relation (2.64), showing that the general functional relation holds. In fact, as remarked earlier, this relation implies the recurrence relation itself.

4. q-DIFFERENCE EQUATIONS

Our first result is a q-analogue of Theorem 2.1.

THEOREM 4.1. *If $w(z)$ is analytic in the ring $q<|z|<1$ and is continuous on its boundary then*

$$\begin{aligned}(D_q\phi_n)(z)=&\frac{\kappa_{n-1}}{\kappa_n}\frac{1-q^n}{1-q}\phi_{n-1}(z)\\&-i\phi_n^*(z)\int_{|\zeta|=1}\frac{u(\zeta)-u(qz)}{\zeta-qz}\phi_n(\zeta)\,\overline{\phi_n^*(q\zeta)}\,w(\zeta)\,d\zeta\\&+i\phi_n(z)\int_{|\zeta|=1}\frac{u(\zeta)-u(qz)}{\zeta-qz}\phi_n(\zeta)\,\overline{\phi_n(q\zeta)}\,w(\zeta)\,d\zeta.\end{aligned}\tag{4.1}$$

Proof. Expand $D_q\phi_n(z)$ in a series of the ϕ_n's. We get

$$(1-q)(D_q\phi_n)(z)=\int_{|\zeta|=1}\sum_{k=0}^{n-1}\phi_k(z)\,\overline{\phi_k(\zeta)}\,[\phi_n(\zeta)-\phi_n(q\zeta)]\,w(\zeta)\frac{d\zeta}{i\zeta^2}.$$

Break the above integral as a difference of two integrals involving $\phi_n(\zeta)$ and $\phi_n(q\zeta)$, then in the second integral replace ζ by ζ/q. Under such transformation $\overline{\phi_k(\zeta)}$ is transformed to $\overline{\phi_k(q\zeta)}$, since $|\zeta|=1$. Furthermore (1.4) gives

$$w(\zeta/q)=[1+\zeta u(\zeta)(1-1/q)]\,w(\zeta).\tag{4.2}$$

Therefore

$$\begin{aligned}&(1-q)(D_q\phi_n)(z)\\&=\int_{|\zeta|=1}\sum_{k=0}^{n-1}\phi_k(z)\,\overline{\zeta\phi_k(\zeta)}\,\phi_n(\zeta)\,w(\zeta)\frac{d\zeta}{i\zeta}\\&\quad+\int_{|\zeta|=1}\sum_{k=0}^{n-1}\phi_k(z)[-q\overline{\zeta\phi_k(q\zeta)}+u(\zeta)(1-q)\,\overline{\phi_k(q\zeta)}]\,\phi_n(\zeta)\,w(\zeta)\frac{d\zeta}{i\zeta}\\&=\frac{\kappa_{n-1}}{\kappa_n}\phi_{n-1}(z)-q^n\frac{\kappa_{n-1}}{\kappa_n}\phi_{n-1}(z)\\&\quad+(1-q)\int_{|\zeta|=1}\phi_n(\zeta)\,u(\zeta)\sum_{k=0}^{n-1}\phi_k(z)\,\overline{\phi_k(q\zeta)}\,w(\zeta)\frac{d\zeta}{i\zeta}.\end{aligned}$$

The result now follows from (2.9).

We next substitute for $\phi_n^*(z)$ in (4.2) from (4.3), if $\phi_n(0) \neq 0$, and establish

$$(D_q \phi_n)(z) = A_n(z)\, \phi_{n-1}(z) - B_n(z)\, \phi_n(z), \tag{4.3}$$

with

$$A_n(z) = \frac{\kappa_{n-1}}{\kappa_n} \frac{1-q^n}{1-q} + i \frac{\kappa_{n-1}}{\phi_n(0)} z \int_{|\zeta|=1} \frac{u(\zeta)-u(qz)}{\zeta - qz} \phi_n(\zeta)\, \overline{\phi_n^*(q\zeta)}\, w(\zeta)\, d\zeta \tag{4.4}$$

$$B_n(z) = -i \int_{|\zeta|=1} \frac{u(\zeta)-u(qz)}{\zeta - qz} \phi_n(\zeta) \left[\overline{\phi_n(q\zeta)} - \frac{\kappa_n}{\phi_n(0)} \overline{\phi_n^*(q\zeta)} \right] w(\zeta)\, d\zeta. \tag{4.5}$$

The are the q-analogues of (2.13–2.18). Here again we set

$$L_{n,1} = D_q + B_n(z), \tag{4.6}$$

$$L_{n,2} = -D_q - B_{n-1}(z) + \frac{A_{n-1}(z)\, \kappa_{n-1}}{z \kappa_{n-2}} + \frac{A_{n-1}(z)\, \kappa_n \phi_{n-1}(0)}{\kappa_{n-2} \phi_n(0)}. \tag{4.7}$$

The ladder operations are

$$L_{n,1} \phi_n(z) = A_n(z)\, \phi_{n-1}(z), \qquad L_{n,2} \phi_{n-1}(z) = \frac{\phi_{n-1}(0)\, \kappa_{n-1}}{\phi_n(0)\, \kappa_{n-2}} \frac{A_{n-1}(z)}{z} \phi_n(z). \tag{4.8}$$

This results in the q-difference equation

$$L_{n,2} \left(\frac{1}{A_n(z)} L_{n,1} \right) \phi_n(z) = \frac{A_{n-1}(z)}{z} \frac{\phi_{n-1}(0)\, \kappa_{n-1}}{\phi_n(0)\, \kappa_{n-2}} \phi_n(z). \tag{4.9}$$

There is also a q-analogue of the functional equation (3.2) which can be found most simply by exploiting the third Remark to Theorem (3.1).

THEOREM 4.2. *If $u(z)$ is analytic in the annular region $q < |z| < 1$ then the following functional equation for the coefficients $A_n(z)$, $B_n(z)$ holds*

$$B_n + B_{n-1} - \frac{\kappa_{n-1}}{\kappa_{n-2}} \frac{A_{n-1}}{z} - \frac{\kappa_n}{\kappa_{n-2}} \frac{\phi_{n-1}(0)}{\phi_n(0)} A_{n-1} = -\frac{n-1}{qz} - \frac{u(qz)}{q} - \frac{1-q}{q} \sum_{j=0}^{n-1} \left[B_{j+1} - \frac{\kappa_j}{\kappa_{j-1}} \frac{A_j}{z} \right]. \tag{4.10}$$

Proof. Two alternative forms of the second order q-difference equation are possible, namely (4.9) and the following,

$$L_{n+1,1} \left(\frac{z}{A_n(z)} L_{n+1,2} \right) \phi_n(z) = \frac{\kappa_n \phi_n(0)}{\kappa_{n-1} \phi_{n+1}(0)} A_{n+1}(z)\, \phi_n(z). \tag{4.11}$$

These two equations, written out in full are, respectively

$$D_q^2\phi_n(z)+\left\{B_n(qz)+\frac{A_n(qz)}{A_n(z)}\,B_{n-1}(z)-\frac{D_qA_n(z)}{A_n(z)}\right.$$
$$-\frac{\kappa_{n-1}}{\kappa_{n-2}}\frac{A_n(qz)\,A_{n-1}(z)}{A_n(z)\,z}-\frac{\kappa_n}{\kappa_{n-2}}\frac{\phi_{n-1}(0)}{\phi_n(0)}\frac{A_n(qz)\,A_{n-1}(z)}{A_n(z)}\Big\}\,D_q\phi_n(z)$$
$$+\left\{D_qB_n(z)-\frac{B_n(z)}{A_n(z)}\,D_qA_n(z)+\frac{A_n(qz)}{A_n(z)}\,B_n(z)\,B_{n-1}(z)\right.$$
$$-\frac{\kappa_{n-1}}{\kappa_{n-2}}\frac{A_n(qz)}{A_n(z)}\frac{A_{n-1}(z)\,B_n(z)}{z}-\frac{\kappa_n}{\kappa_{n-2}}\frac{\phi_{n-1}(0)}{\phi_n(0)}\frac{A_n(qz)}{A_n(z)}\,A_{n-1}(z)\,B_n(z)$$
$$+\frac{\kappa_{n-1}}{\kappa_{n-2}}\frac{\phi_{n-1}(0)}{\phi_n(0)}\frac{A_n(qz)\,A_{n-1}(z)}{z}\Big\}\,\phi_n(z)=0, \tag{4.12}$$

and

$$D_q^2\phi_n(z)+\left\{\frac{A_n(qz)}{qA_n(z)}\,B_{n+1}(z)+B_n(qz)-\frac{D_qA_n(z)}{qA_n(z)}\right.$$
$$-\frac{\kappa_n}{\kappa_{n-1}}\frac{A_n(qz)}{qz}-\frac{\kappa_{n+1}}{\kappa_{n-1}}\frac{\phi_n(0)}{\phi_{n+1}(0)}\,A_n(qz)+\frac{1}{qz}\Big\}\,D_q\phi_n(z)$$
$$+\left\{D_qB_n(z)-\frac{B_n(z)}{qA_n(z)}\,D_qA_n(z)+\frac{A_n(qz)}{qA_n(z)}\,B_{n+1}(z)\,B_n(z)\right.$$
$$-\frac{\kappa_n}{\kappa_{n-1}}\frac{A_n(qz)\,B_{n+1}(z)}{qz}-\frac{\kappa_{n+1}}{\kappa_{n-1}}\frac{\phi_n(0)}{\phi_{n+1}(0)}\frac{A_n(qz)\,B_{n+1}(z)}{q}$$
$$+\frac{\kappa_n}{\kappa_{n-1}}\frac{\phi_n(0)}{\phi_{n+1}(0)}\frac{A_n(qz)\,A_{n+1}(z)}{qz}$$
$$+\frac{B_n(z)}{qz}-\frac{\kappa_{n+1}}{\kappa_{n-1}}\frac{\phi_n(0)}{\phi_{n+1}(0)}\frac{A_n(qz)}{qz}\Big\}\,\phi_n(z)=0. \tag{4.13}$$

A comparison of the coefficients of the first q-difference terms leads to the difference equation

$$\frac{1}{q}\,B_{n+1}(z)-B_{n-1}(z)-\frac{\kappa_n}{\kappa_{n-1}}\frac{A_n(z)}{qz}+\frac{\kappa_{n-1}}{\kappa_{n-2}}\frac{A_{n-1}(z)}{z}$$
$$-\frac{\kappa_{n+1}}{\kappa_{n-1}}\frac{\phi_n(0)}{\phi_{n+1}(0)}\,A_n(z)+\frac{\kappa_n}{\kappa_{n-2}}\frac{\phi_{n-1}(0)}{\phi_n(0)}\,A_{n-1}(z)=-\frac{1}{qz}. \tag{4.14}$$

Using the results for the first coefficients

$$B_1(z)=-u(qz)-\frac{\phi_1(qz)}{\phi_1(0)}\,M_1(z), \tag{4.15}$$

$$\frac{A_0(z)}{\kappa_{-1}}=-zM_1(z), \tag{4.16}$$

with

$$M_1(z) \equiv \int_{|\zeta|=1} \zeta \frac{u(\zeta)-u(qz)}{\zeta - qz} w(\zeta) \frac{d\zeta}{i\zeta}, \tag{4.17}$$

this difference equation can be summed to yield the result in (4.10).

In the next example we will follow the notation and terminology in [9] and [1]. The q-shifted factorials are

$$(a; q)_0 := 1, \qquad (a; q)_n := \prod_{k=1}^{n} (1 - aq^{k-1}), \qquad n = 1, \cdots \text{ or } \infty,$$

while the multi-shifted factorials are

$$(a_1, a_2, ..., a_m; q)_n := \prod_{k=1}^{m} (a_k; q)_n.$$

Define the inner product

$$(f, g) = \int_{|z|=1} f(\zeta)\, \overline{g(\zeta)}\, w(\zeta) \frac{d\zeta}{\zeta}. \tag{4.18}$$

With respect to this inner product the adjoint of $L_{n,1}$ is

$$(L^*_{n,1} f)(z) = z^2[q - (1-q)\, z\overline{u(z)}]\, D_q f(z) + zf(z) + [\overline{B_n(z)} + \overline{u(z)}]\, f(z), \tag{4.19}$$

provided that $w(z)$ is analytic in $q < |z| < 1$ and is continuous on $|z| = 1$ and $|z| = q$. The proof follows from the definition of D_q and the fact $\overline{g(\zeta)} = \bar{g}(1/\zeta)$, when $|\zeta| = 1$. Observe that as $q \to 1$, the right-hand side of (4.19) tends to the right-hand side of (2.25), as expected.

EXAMPLE. Consider the Rogers–Szegő [32] polynomials $\{\mathscr{H}_n(z \mid q)\}$, where

$$w(z) = \frac{(q^{1/2}z, q^{1/2}/z; q)_\infty}{2\pi (q; q)_\infty}, \tag{4.20}$$

$$\mathscr{H}_n(z \mid q) = \sum_{k=0}^{n} \frac{(q; q)_n\, q^{-k/2} z^k}{(q; q)_k\, (q; q)_{n-k}}.$$

In this case

$$\phi_n(z)=\frac{q^{n/2}}{\sqrt{(q;q)_n}}\,\mathscr{H}_n(z\mid q),\ \phi_n(0)=\frac{q^{n/2}}{\sqrt{(q;q)_n}},\ \kappa_n=\frac{1}{\sqrt{(q;q)_n}}. \quad (4.21)$$

It is easy to see that

$$u(z)=\frac{\sqrt{q}}{1-q}+\frac{qz^{-1}}{1-q}. \quad (4.22)$$

Thus $[u(\zeta)-u(qz)]/(\zeta-qz)$ is $-1/[(1-q)\,\zeta z]$. A simple calculation gives

$$(D_q\phi_n)(z)=\frac{(1-q^n)^{3/2}}{1-q}\,\phi_{n-1}(z)$$
$$+\frac{\kappa_{n-1}\phi_{n-1}(z)}{\phi_n(0)(1-q)}\int_{|\zeta|=1}\phi_n(\zeta)\,\overline{\phi_n^*(q\zeta)}\,w(\zeta)\,\frac{d\zeta}{i\zeta},$$

which simplifies to

$$(D_q\phi_n)(z)=\frac{\sqrt{1-q^n}}{1-q}\,\phi_{n-1}(z), \quad (4.23)$$

since $\kappa_n\overline{\phi_n^*(q\zeta)}-\phi_n(0)\,\overline{\phi_n(q\zeta)}$ is a polynomial of degree $n-1$. The functional equation (4.23) can be verified independently by direct computation.

5. DISCRIMINANTS

Schur [28], [33, Sect. 6.71] gave an interesting proof of the Stieltjes–Hilbert evaluation of the discriminants of the classical orthogonal polynomials of Hermite, Laguerre, and Jacobi. His proof relies on a very clever observation. Let $\{p_n(x)\}$ be a sequence of polynomials satisfying a three term recurrence relation

$$p_n(x)=(a_nx+b_n)\,p_{n-1}(x)-c_np_{n-2}(x),\qquad n>1, \quad (5.1)$$

and the initial conditions

$$p_0(x)=1;\qquad p_1(x)=a_1x+b_1, \quad (5.2)$$

together with the conditions $a_{n-1}c_n \neq 0, n>1$. Schur [28] observed that

$$\prod_{k=1}^{n} p_{n-1}(x_{j,n}) = (-1)^{n(n-1)/2} \prod_{k=1}^{n} a_k^{n-2k+1} c_k^{k-1}, \tag{5.3}$$

where $\{x_{j,n}: 1 \leqslant j \leqslant n\}$ is the set of zeros of $p_n(x)$.

Let $z_{j,n}$ be the zeros of $\phi_n(z)$. Following Schur, we let

$$\Delta_n = \prod_{j=1}^{n} \phi_{n-1}(z_{j,n}). \tag{5.4}$$

LEMMA 5.1. *The expression Δ_n is given by*

$$\Delta_n = \frac{[\phi_n(0)]^{n-1}}{\kappa_n^{n-1}\kappa_{n-1}^n} \prod_{j=1}^{n-1} \kappa_j^2, \qquad n \geqslant 2, \Delta_1 = 1. \tag{5.5}$$

Proof. It is clear that

$$\begin{aligned} \Delta_n &= \kappa_{n-1}^n \prod_{j=1}^{n} \prod_{k=1}^{n-1} (z_{j,n} - z_{k,n-1}) \\ &= \frac{\kappa_{n-1}^n}{k_n^{n-1}} \prod_{k=1}^{n-1} \phi_n(z_{k,n-1}) \end{aligned} \tag{5.6}$$

The recurrence relation (2.5) we find

$$\phi_n(z_{k,n-1}) = -\frac{\kappa_{n-2}\phi_n(0)}{\kappa_{n-1}\phi_{n-1}(0)} z_{k,n-1}\phi_{n-2}(z_{k,n-1}). \tag{5.7}$$

Substituting from (5.7) into (5.6) and applying $\phi_{n-1}(0) = \kappa_{n-1} \prod_{j=1}^{n-1} (-z_{j,n-1})$ we establish the two term recurrence relation

$$\Delta_n = \frac{\kappa_{n-2}^{n-1}[\phi_n(0)]^{n-1}}{\kappa_n^{n-1}[\phi_{n-1}(0)]^{n-2}} \Delta_{n-1}, \qquad n>1.$$

By direct computation we find $\Delta_1 = 1$, so the above two term recursion implies (5.5).

EXAMPLE. For the circular Jacobi polynomials the discriminant is given by

$$\Delta_n = \left(\frac{a}{n+a}\right)^{n-1} \left[\frac{(n-1)!\,(2a+1)_{n-1}}{(a+1)_{n-1}^2}\right]^{n/2} \prod_{j=1}^{n-1} \frac{(a+1)_j^2}{j!\,(2a+1)_j}, \tag{5.8}$$

while those for the Szegő polynomials are

$$\Delta_{2n}=\left(\frac{a+b}{2n+a+b}\right)^{2n-1}\left[\frac{(n-1)!\,(a+b+1)_{n-1}\,(a+1/2)_n\,(b+1/2)_n}{(a+b+1)_{2n-1}^2}\right]^n \times\frac{1}{(a+1/2)_n\,(b+1/2)_n}\left[\frac{\prod_{j=1}^{2n-1}(a+b+1)_j}{\prod_{l=1}^{n-1} l!\,(a+b+1)_l\,(a+1/2)_l\,(b+1/2)_l}\right]^2, \tag{5.9}$$

$$\begin{aligned}\Delta_{2n-1}=&\left(\frac{a-b}{2n-1+a+b}\right)^{2n-2}\\ &\times\left[\frac{(n-1)!\,(a+b+1)_{n-1}\,(a+1/2)_{n-1}(b+1/2)_{n-1}}{(a+b+1)_{2n-2}^2}\right]^{n-1/2}\\ &\times(n-1)!\,(a+b+1)_{n-1}\\ &\times\left[\frac{\prod_{j=1}^{2n-2}(a+b+1)_j}{\prod_{l=1}^{n-1} l!\,(a+b+1)_l\,(a+1/2)_l\,(b+1/2)_l}\right]^2.\end{aligned} \tag{5.10}$$

The resultant of two polynomials f_n and g_m is

$$R\{f_n, g_m\}=\gamma^m\prod_{j=1}^{n} g_m(z_j), \tag{5.11}$$

where f_n is as in (1.7). Observe that [6, Sect. 100]

$$D(f_n)=(-1)^{n(n-1)/2}\,\gamma^{-1}\,R\{f_n, f_n'\}. \tag{5.12}$$

In general let T be a degree reducing operator T, that is $(Tf)(x)$ is a polynomial of exact degree $n-1$ when f has precise degree n and the leading terms in f and Tf have the same sign. Define the generalized discriminant $D(f_n, T)$ by

$$\begin{aligned}D(f_n, T):&=(-1)^{n(n-1)/2}\,\gamma^{-1}R\{f_n, Tf_n\}\\ &=(-1)^{n(n-1)/2}\,\gamma^{n-2}\prod_{j=1}^{n}(Tf_n)(z_j),\end{aligned} \tag{5.13}$$

for f_n as in (1.7).

THEOREM 5.2. *Let $\{\phi_n(z)\}$ be orthonormal on the unit circle and assume that T is a linear operator such that*

$$T\phi_n(z)=A_n(z)\,\phi_{n-1}(z)-B_n(z)\,\phi_n(z). \tag{5.14}$$

Let $\{z_{k,n}: 1 \leqslant k \leqslant n\}$ *be the zeros of* $\phi_n(z)$. *Then the generalized discriminant* (5.13) *is given by*

$$D(\phi_n, T) = \frac{(-1)^{n(n-1)/2} [\phi_n(0)]^{n-1}}{\kappa_n \kappa_{n-1}^n} \prod_{j=1}^{n-1} \kappa_j^2 \prod_{k=1}^{n} A_n(z_{k,n}), \qquad n > 0. \tag{5.15}$$

Proof. Apply (5.14), (5.4) and (5.5).

In the case of the orthonormal Rogers–Szegő polynomials Theorem 4.2 and, (4.21) and (4.23) imply the discriminant formula

$$D(\phi_n, D_q) = D(\phi_n, q) = \frac{(-q)^{n(n-1)/2}}{(1-q)^n} (q; q)_n \prod_{j=1}^{n-1} \left[\frac{1}{(q; q)_j} \right]. \tag{5.16}$$

For the Rogers–Szegő polynomials we get

$$D(\mathscr{H}_n, q) = (-q)^{-n(n-1)/2} \left[\frac{(q; q)_n}{(1-q)} \right]^n \prod_{j=1}^{n-1} \left[\frac{1}{(q; q)_j} \right]. \tag{5.17}$$

If one is interested in the limiting case $q \to 1$ then we need to rewrite (5.17) as

$$D(\mathscr{H}_n, q) = (-q)^{n(n-1)/2} (1-q)^{n(n-1)/2} \left[\frac{(q; q)_n}{(1-q)^n} \right]^n \prod_{j=1}^{n} \left[\frac{(1-q)^j}{(q; q)_j} \right], \tag{5.18}$$

which shows that $D(\mathscr{H}_n, q) \to 0$, for $n > 1$, when $q \to 1$, as expected since $\mathscr{H}_n(z \mid q) \to (1+z)^n$ as $q \to 1$.

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